On the Levi problem with singularities

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1. Introduction

Is a complex space X which is the union of an increasing sequence $X_1 \subset X_2 \subset X_3 \subset \cdots$ of open Stein subspaces itself a Stein space ?

From the begining this question has held great interest in Stein theory.

The special case when $\{X_j\}_{j\geq 1}$ is a sequence of Stein domains in $\mathbb{C}^n$ had been proved long time ago by Behnke and Stein [2].

In 1956, Stein [13] answered positively the question under the additional hypothesis that X is reduced and every pair $(X_{\nu+1}, X_\nu)$ is Runge.

In the general case X is not necessarily holomorphically-convex. Fornaess [7], gave a 3-dimensional example of such situation.

In 1977, Markoe [10] proved the following:
Let X be a reduced complex space which the union of an increasing sequence $X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$ of Stein domains.

Then X is Stein if and only if $H^1(X, O_X) = 0$.

M. Coltoiu has shown in [3] that if $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$ is an increasing sequence of Stein domains in a normal Stein space X, then $D = \bigcup_{j\geq 1} D_j$ is a domain of holomorphy. (i.e. for each $x \in \partial D$ there is $f \in O(D)$ which is not holomorphically extendable through x).

The aim of this paper is to prove the following theorems:

theorem 1 *-Let X be a Stein normal space of dimension n and $D \subset\subset X$ an open subset which the union of an increasing sequence $D_1 \subset D_2 \subset \cdots \subset D_n \cdots$ of domains of holomorphy in X. Then D is a domain of holomorphy.*

theorem 2 -*A domain of holomorphy D which is relatively compact in a 2-dimensional normal Stein space X itself is Stein*

theorem 3 -*Let X be a Stein space of dimensinon n and $D \subset X$ an open subspace which is the union of an increasing sequence $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$ of open Stein subsets of X. Then D itself is Stein, if X has isolated singularities.*

2. Preliminaries

It should be remarked that the statement of theorem 2 is in general false if $dim(X) \geq 3$:
Let $X = \{z \in \mathbb{C}^4 : z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}, H = \{z \in \mathbb{C}^4; z_1 = iz_2, z_3 = iz_4\}$, $U = \{z \in X : |z| < 1\}$, and $D = U - U \cap H$.
X is a Stein normal space of dimension 3 with the singularity only at the origine. Since D is the complement of a hypersurface on the Stein space U, then D is a domain of holomorphy. But D is not Stein. (See [9]).

Let X be a connected n-dimensional, Stein normal space and Y be the singular locus of X.
There exist finitely many holomorphic maps

$$\phi_j : X \longrightarrow \mathbb{C}^n, j = 1, \cdots, l$$

with discrete fibers, and holomorphic functions $f_1, \cdots, f_l$ on X such that the branch locus of ϕ_j is contained in $Z_j = \{f_j = 0\}$ and $Y = \bigcap_{j=1}^{l} Z_j$.

3. Proofs of theorems

We prove theorem 1 using the method of Fornaess and Narasimhan [8] (See also lemma 7, [1]).
For every irreducible component X_i of X, $X_i \cap D \subset\subset X_i$ is an irreducible component of D and a union of an increasing sequence of domains of holomorphy in the Stein space X_i. (See [11]).
Since each X_i is normal and $(X_i \cap D)_i$ are pairwise disjoint domains, then we may assume that X is connected.

Let $q \in \partial D - Y$ and choose holomorphic functions $h_1, \cdots, h_m$ on X such that: $\{x \in X / h_i(x) = 0, i = 1, \cdots, m\} = \{q\}$, and j such that $q \notin Z_j$.
Since $D - Z_j$ is the union of the increasing sequence $(D_k - Z_j)_{k \geq 1}$ of the Stein sets $D_k - Z_j$ in the Stein manifold $X - Z_j$, then $D - Z_j$ is Stein.

Let d_j be the boundary distance of the unramified domain $\phi_j : D - Z_j \longrightarrow \mathbb{C}^n$. Then $-logd_j$ is plurisubharmonic on $D - Z_j$. Therefore the function

$$\psi_j(z) = \begin{cases} Max(0, -logd_j + k_j log|f_j|) & on \quad D - Z_j \\ 0 \quad on \quad Z_j \end{cases}$$

is plurisubharmonic on D, if k_j is a large constant. This follows from a result due to Oka. (See also [1], lemma 7).

By the Nullstellensatz, There exist a neighborhood V of q in X and constants $c > 0, N > 0$ such that

$$\sum_{i=1}^{m} |h_i(x)|^2 \geq c|\phi_j(x) - \phi_j(q)|^N, x \in V$$

Since ϕ_j is an analytic isomorphism at q and $f_j(q) \mp 0$, it follows that, if V is sufficiently small, there is a constant $c_0 > 0$ such that

$$\sum_{i=1}^{m} |h_i(x)|^2 \geq c_0 exp(-N\psi_j(x)), x \in V \cap D.$$

Now, since $\psi_j \geq 0$, there exist constants $c_1, c_2 > 0$ such that

$$c_2 exp(-N\psi_j(x)) \leq \sum_{i=1}^{m} |h_i(x)|^2 \leq c_1 exp(\psi_j(x)), x \in D.$$

And applying the theorem of Skoda [14], we deduce that there is a constant $k > 0$ and holomorphic functions $g_1, \cdots, g_m$ on $D - Z_j$ such that

$$\sum_{i=1}^{m} g_i h_i = 1 \quad on \quad D - Z_j$$

and

$$\sum_{i=1}^{m} \int_{D - Z_j} |g_i(x)|^2 exp(-k\psi_j(x)) dv < \infty$$

Where dv is Lebesgue measure pulled back to D and k depending only on N and m. The existence of a holomorphic function f on D which is unbounded on any sequence $\{q_\mu\}$ of points approaching q follows from lemma 3-1-2 of Fornaess-Narasimhan [8]. Since $\partial D - Y$ is dense in ∂D, it follows that D is a domain of holomorphy.

We shall prove theorem 2 using the following result of R.Simha [15].

theorem 4 *-Let X be a normal Stein complex space of dimension 2, and H a hypersurface in X. Then $X - H$ is Stein.*

Proof of theorem 2

By the theorem of Andreotti-Narasimhan [1], it is suffficient to prove that D is locally Stein, and we may of course assume that X is connected.

Let $p \in \partial D \cap Y$, and choose a connected Stein open neighborhood U of p with $U \cap Y = \{p\}$ and such that U is biholomorphic to a closed analytic set of a domain M in some $\mathbb{C}^N$. Let E be a complex affine subspace of $\mathbb{C}^N$ of maximal dimension such that p is an isolated point of $E \cap U$.

By a coordinate transformation one can obtain that $z_i(p) = 0$ for all $i \in \{1, \cdots, N\}$ and we may assume that there is a connected Stein neighborhood V of p in M such that $U \cap V \cap \{z_1(x) = z_2(x) = 0\} = \{p\}$.

We may, of course, suppose that $N \geq 4$, and let $E_1 = V \cap \{z_1(x) = \cdots = z_{N-2}(x) = 0\}$, $E_2 = \{x \in E_1 : z_{N-1}(x) = 0\}$. Then $A = (U \cap V) \cup E_1$ is a Stein closed analytic set in V as the union of two Stein global branches of A.

Let $\zeta : \hat{A} \to A$ be a normalization of A. Then $\zeta : \hat{A} - \zeta^{-1}(p) \to A - \{p\}$ is biholomorphic. Since $\zeta^{-1}(E_1) = \{x \in \hat{A} : z_1(\zeta(x)) = \cdots = z_{N-2}(\zeta(x)) = 0\}$ is everywhere 1-dimensional, it follows from theorem 4 that $\hat{A} - \zeta^{-1}(E_2)$ is Stein. Hence $A - E_2 = \zeta(\hat{A} - \zeta^{-1}(E_2))$ itself is Stein.

Since $p \in E_2$ is the unique singular point of A, then $U \cap V \cap D$ is Stein being a domain of holomorphy in the Stein manifold $A - E_2$.

If $p \in \partial D - Y$, then there exists j such that $p \notin Z_j$. We can find a Stein open neighborhood U of p in X such that $U \cap Z_j = \emptyset$. Then $U \cap D = U \cap (D - Z_j)$ is Stein.

The main step in the proof of theorem 3 is to show, when D is, in addition, relatively compact in X, that for all $p \in \partial D$, there exist an open neighborhood U of p in X and an exhaustion function f on $D \cap U$ such that for each open $V \subset\subset U$ there is a continuous function g on V which is locally the maximum of a finite number of stricly plurisubharmonic functions with $|f - g| < 1$. Which implies that $D \cap U$ is 1-complete with corners.
(A result due to Peternell [12]).
This result will be applied in connection with
the Diederich-Fornaess theorem [6] which asserts that an irreducible n-dimensional complex space X is Stein if X is 1-complete with corners.
The proof is also based on the following result of M.Peternell [12]

lemma 1 *-Let X be a complex space of pure dimension n,* $W \subset X \times X$ *be an open set and* $f \in F_n(W - \Delta_X)$ *where* $\Delta_X = \{(x, x) : x \in X\}$*, and let* $S \subset\subset S' \subset\subset X$ *be open subsets of X such that* $S \times S' \subset\subset W$*. Define* $s(x) = Sup\{f(x, y) : y \in \overline{S'} - S\}$ *for* $x \in S$ *and assume that* $s(x) > f(x, y)$ *if* $y \in \partial S'$*.*
If S is Stein, then for each $D \subset\subset S$ *and each* $\varepsilon > 0$*, there is a* $g \in F_1(D)$ *such that* $|g - s| < \varepsilon$ *on D.*
Here $F_n(D)$ *and* $F_1(D)$ *denote respectively the sets of continuous functions on D which are locally the maximum of a finite number of strongly n-convex (resp. stricly psh) functions.*

Proof of theorem 3

Clearly we may suppose that D is relatively compact in X.
Since the Stein property is invariant under normalization [11], we may assume that X is normal and connected.
For $n = 2$, theorem 3 follows as an immediate consequence of theorem 2. Then we may also assume that $n \geq 3$.
Let $p \in \partial D \cap Y$, and choose a Stein open neighborhood U of p in X that can be realized as a closed complex subspace of a domain M in $\mathbb{C}^N$.
Let E be a complex affine subspace of $\mathbb{C}^N$ of maximal dimension such that p is an isolated point of $E \cap U$, and let E' be any complementary complex affine subspace to E in $\mathbb{C}^N$ through p.

We may choose the coordinates $z_1, \cdots, z_N$ and the space E' such that $z_i(p) = 0$ for all $i \in \{1, \cdots, N\}$ and $dim(E' \cap U) \geq 1$. Since $T = E' \cap U$ is a closed analytic set in U, and $h(z,w) = |z|^2 + |w|^2 - log(|z-w|^2)$ a strongly n-convex C^∞ function on $E' \times E' - \Delta_{E'}$, then there exists a strongly n-convex C^∞ function ψ on a neighborhood W of $T' = T \times T - \Delta_T$ with $W \subset U \times U - \Delta_U$ such that $h \leq \psi/_{T'} \leq h + 1$. (See Demailly [5]).

Let W' be an open set in $U \times U$ such that $W = W' - \Delta_U$. We may choose W'such that there exist a neighborhood N of p in X and a Stein open neighborhood U_1 of p with $U_1 \subset\subset N$ and such that $U_1 \times (N - U_1) \subset\subset W'$.

We now construct an exhaustion function f_1 on $U_1 \cap D$ such that for each open $Z \subset\subset U_1 \cap D$ there is a $g \in F_1(Z)$ with $|g - f_1| < 1$.

Let $f_1(z) = Sup\{\psi(z,w), w \in \overline{N} - U_1 \cap D\}$, $z \in U_1 \cap D$

Obviously f_1 is an exhaustion function on $U_1 \cap D$. There exists $m \geq 1$ such that $Z \subset\subset U_1 \cap D_m$.

We now define

$$g_j(z) = Sup\{\psi(z,w) : w \in \overline{N} - U_1 \cap D_j\},\ for\ z \in U_1 \cap D_j,\ j \geq m$$

Since $U_1 \cap D_j$ is Stein, $\psi(z,w)$ is n-convex on W, and $g_j(z) > \psi(z,w)$ for every $(z,w) \in (U_1 \cap D_j) \times \partial N$, then there is a $h_j \in F_1(Z)$ such that $|g_j - h_j| < \frac{1}{2}$. Since, obviously, $(g_j)_{j \geq 1}$ converges uniformally on compact sets to f_1, then there is a $j \geq m$ such that $|g_j - f_1| < \frac{1}{2}$ on Z. Hence $|f_1 - h_j| < 1$. Now the theorem follows from the lemma and the theorem of Diederich-Fornaess [6]

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